

# INDEPENDENT AND FREE SETS IN UNIVERSAL ALGEBRAS

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**ABSTRACT.** We discuss notions of independent and free sets in universal algebras, and more generally, in spaces endowed with a hull operator. Our main result says that for each universal algebra  $\mathbb{A} = (A, \mathcal{A})$  of cardinality  $|A| \geq 2$  and an infinite set  $X$  of cardinality  $|X| \geq |\mathcal{A}|$ , the  $X$ -power  $\mathbb{A}^X = (A^X, \mathcal{A}^X)$  of the algebra  $\mathbb{A}$  contains an  $\mathcal{A}$ -free subset  $\mathcal{F} \subset A^X$  of cardinality  $|\mathcal{F}| = 2^{|X|}$ . This generalizes a classical Fichtenholtz-Katorovitch-Hausdorff result on the existence of large independent families of subsets of a given infinite set.

## INTRODUCTION

It is well-known that each maximal linearly independent subset of a linear space  $X$  generates  $X$ . A natural notion of independence can be also defined for other algebraic structures, in particular, for universal algebras. Unfortunately, even for relative simple universal algebras (like groups) the notion of independence does not work as good as for linear spaces. For example, any infinite linear space  $V$  over the two-element field  $\mathbb{F}_2 = \{0, 1\}$  can be embedded into a non-commutative group  $G$  of cardinality  $|G| = 2^{|V|}$  so that each maximal linearly independent subset  $B \subset V$  remains maximal independent in  $G$  and hence does not generate the whole group  $G$ . This fact (proved in Proposition 1.3) shows that the problem of constructing large independent sets in general algebraic setting is not trivial.

A general notion of algebraic independence was actually introduced by Marczewski in [7] and studied in subsequent papers [8], [9]. A subset  $B$  of an abstract algebra  $(A, \mathcal{A})$  is called ([7]) independent provided each function from  $B$  in  $A$  can be extended to a homomorphism defined on the algebra generated by  $B$ . This notion is a generalization of the linear and algebraic independence of numbers, linear independence of vectors, independence of sets and elements of Boolean algebras and many others. In our paper, Marczewski's independent sets will be called "free". The notion of a "hull" (or "closure") was introduced by Birkhoff [3] and also considered by Marczewski (e.g. in [8]). Other concepts of independence considered in our paper can be also found in [8]. For the reader's convenience we recall some proofs and considerations in the modern notation.

Our main result is Theorem 4.3 saying that for each universal algebra  $\mathbb{A} = (A, \mathcal{A})$  of cardinality  $|A| \geq 2$  and each infinite set  $X$  of cardinality  $|X| \geq |\mathcal{A}|$ , the  $X$ -power  $\mathbb{A}^X = (A^X, \mathcal{A}^X)$  of the algebra  $\mathbb{A}$  contains an independent set  $\mathcal{F} \subset A^X$  of cardinality  $|\mathcal{F}| = 2^{|X|}$ . This generalizes a classical Fichtenholtz-Katorovitch-Hausdorff result on the existence of an independent family  $\mathcal{F} \subset \mathcal{P}(X)$  of cardinality  $|\mathcal{F}| = 2^{|X|}$  in the power-set  $\mathcal{P}(X)$  of  $X$ . The other famous generalization of Fichtenholtz-Katorovitch-Hausdorff theorem is the Balcar-Franěk theorem on the number of independent elements in complete Boolean algebras [1].

## 1. HULL OPERATORS AND (STRONGLY) INDEPENDENT SETS

By a *hull operator* on a set  $X$  we understand a function  $H : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  defined on a family of all subsets of  $X$ , which is *monotone* in the sense that for any subsets  $A \subseteq B \subseteq X$  we get  $A \subseteq H(A) \subseteq H(B) \subseteq X$ .

Let  $H : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  be a hull operator on a set  $X$ . A subset  $B \subset X$  is called

- *H-independent* if  $b \notin H(B \setminus \{b\})$  for each  $b \in B$ ;
- *strongly H-independent* if  $B \cap H(\emptyset) = \emptyset$  and  $H(B_1) \cap H(B_2) = H(B_1 \cap B_2)$  for any subsets  $B_1, B_2 \subset B$ .

**Proposition 1.1.** *Let  $X$  be a set and  $H$  be a hull operator on  $X$ . Each strongly  $H$ -independent set  $B \subset X$  is  $H$ -independent.*

*Proof.* Assuming that a strongly  $H$ -independent set  $B$  is not  $H$ -independent, we can find a point  $b \in B$  such that  $b \in H(B \setminus \{b\})$ . Then  $b \in H(\{b\}) \cap H(B \setminus \{b\}) = H(\{b\} \cap (B \setminus \{b\})) = H(\emptyset)$ , which is not possible as  $B \subset X \setminus H(\emptyset)$ .  $\square$

Proposition 1.1 can be reversed for hull operators of matroid type. We say that a hull operator  $H : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$

- is *idempotent* if  $H(H(A)) = H(A)$  for each subset  $A \subset X$ ;

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- has *finite supports* if for each subset  $A \subset X$  and a point  $x \in H(A)$  there is a finite subset  $F \subset A$  with  $x \in H(F)$ ;
- has the *MacLane-Steinitz exchange property* if for any subset  $A \subset X$  and points  $x, y \in X \setminus H(A)$  the inclusion  $x \in H(A \cup \{y\})$  is equivalent to  $y \in H(A \cup \{x\})$ ;
- is of *matroid type* if  $H$  is idempotent, has finite supports and has the MacLane-Steinitz exchange property.

It is well-known (and easy to see) that the operator of taking the linear hull in a linear space is of matroid type.

**Proposition 1.2.** *Let  $H : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  be a hull operator of matroid type on a set  $X$ .*

- (1) *A subset  $B \subset X$  is  $H$ -independent if and only if it is strongly  $H$ -independent.*
- (2) *Each  $H$ -independent subset lies in a maximal  $H$ -independent subset of  $X$ .*
- (3) *For each maximal  $H$ -independent subset  $B \subset X$  its hull  $H(B) = X$ .*

*Proof.* This proposition is known for finite sets  $X$ , see [10, §1.4]. For convenience of the reader we present a proof for an arbitrary set  $X$ .

1. Proposition 1.1 implies that each strongly  $H$ -independent subset of  $X$  is  $H$ -independent. Now assume that a set  $B$  is  $H$ -independent. Then for each  $b \in B$  we get  $b \notin H(B \setminus \{b\}) \supset H(\emptyset)$  and hence  $B \cap H(\emptyset) = \emptyset$ . To show that  $B$  is strongly  $H$ -independent, it remains to check that  $H(B_1) \cap H(B_2) \subset H(B_1 \cap B_2)$  for any subsets  $B_1, B_2 \subset B$ . This inclusion is trivial if  $B_1 \subset B_2$  or  $B_2 \subset B_1$ . So, we assume that  $B_1 \not\subset B_2$  and  $B_2 \not\subset B_1$ . Given any point  $h \in H(B_1) \cap H(B_2)$ , we should prove that  $h \in H(B_1 \cap B_2)$ . Assume conversely that  $h \notin H(B_1 \cap B_2)$ . Since the hull operator  $H$  has finite supports, there are minimal finite subsets  $F_1 \subset B_1$  and  $F_2 \subset B_2$  such that  $h \in H(F_1) \cap H(F_2)$ . It follows from  $F_1 \cap F_2 \subset H(F_1 \cap F_2) \subset H(B_1 \cap B_2) \not\ni h$  that  $h \notin F_1 \cap F_2$  and hence  $h \notin F_1$  or  $h \notin F_2$ . Without loss of generality,  $h \notin F_1$ . It follows from  $H(F_1) \cap H(F_2) \not\subset H(F_1 \cap F_2)$ , that  $F_1 \not\subset F_2$  and  $F_2 \not\subset F_1$ . Choose any point  $b \in F_1 \setminus F_2$  and let  $A = F_1 \setminus \{b\}$ . The  $H$ -independence of the set  $B$  implies  $b \notin H(B \setminus \{b\}) \supset H(A)$  and the minimality of the set  $F_1$  implies  $h \notin H(A)$ . Since  $h \in H(F_1) = H(A \cup \{b\})$ , the MacLane-Steinitz exchange property of  $H$  guarantees that

$$b \in H(A \cup \{h\}) \subset H(A \cup H(F_2)) \subset H(H(A \cup F_2)) = H(A \cup F_2) \subset H(B \setminus \{b\}),$$

which contradicts the  $H$ -independence of  $B$ .

2. The second statement can be easily proved using Zorn's Lemma and the fact that the hull operator  $H$  has finite supports.

3. Let  $B$  be a maximal  $H$ -independent subset in  $X$ . If  $H(B) \neq X$ , then we can choose a point  $x \in X \setminus H(B)$ . By the maximality of  $B$  the set  $B_x = B \cup \{x\}$  is not  $H$ -independent. Consequently, there is a point  $b \in B_x$  such that  $b \in H(B_x \setminus \{b\})$ . If  $b \neq x$ , then consider the set  $A = B \setminus \{b\}$  and observe that  $b \in H(B_x \setminus \{b\}) = H(A \cup \{x\})$ . Then the MacLane-Steinitz exchange property implies  $x \in H(A \cup \{b\}) = H(B)$ , which contradicts the choice of  $x$ . So,  $b = x$  and  $x = b \in H(B_x \setminus \{b\}) = H(B)$ , which contradicts the choice of  $x$ .  $\square$

Unfortunately many hull operators that naturally appear in algebra are not of matroid type. The simplest example is the group hull operator. It assigns to each subset  $A$  of a group  $G$  the subgroup  $H(A) \subset G$  generated by  $A$ . Observe that the set  $\{2, 3\}$  is  $H$ -independent in the group of integers  $\mathbb{Z}$  but fails to be strongly  $H$ -independent in  $\mathbb{Z}$ . Each maximal strongly  $H$ -independent subset  $B$  in the group of rational  $\mathbb{Q}$  is a singleton and hence  $H(B) \neq \mathbb{Q}$ . The following proposition yields an example of a (non-commutative) group  $G$  of cardinality continuum containing a maximal  $H$ -independent subset which is countable and hence does not generate the whole group.

**Proposition 1.3.** *Each infinite linear space  $V$  over the two-element field  $\mathbb{F}_2 = \{0, 1\}$  embeds into a group  $G$  of cardinality  $|G| = 2^{|V|}$  such that for the operator of group hull  $H : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$  each maximal strongly  $H$ -independent subset of  $V$  remains maximal  $H$ -independent in  $G$ .*

*Proof.* Since the field  $\mathbb{F}_2$  is finite, the linear space  $V$  has a Hamel basis of cardinality  $|V|$ . Since each permutation of points of the Hamel basis induces a linear automorphism of the linear space  $V$ , the linear automorphism group  $GL(V)$  of  $V$  has cardinality  $|GL(V)| = 2^{|V|}$ .

Let  $G = V \rtimes GL(V)$  be the semidirect product of the groups  $(V, +)$  and  $GL(V)$ . Elements of the group  $G$  are ordered pairs  $(v, f) \in V \times GL(V)$  and for two pairs  $(v, f), (u, g) \in V \rtimes GL(V)$  their product is defined by the formula

$$(v, f) \cdot (u, g) = (v + f(u), f \circ g).$$

The inverse element to a pair  $(v, f) \in G$  is the pair  $(-f^{-1}(v), f^{-1})$ .

Identify the group  $V$  with the normal subgroup  $V \times \{\text{id}_V\}$  of  $G$ . Here by  $\text{id}_V : V \rightarrow V$  we denote the identity automorphism of  $V$ .

Let  $M \subset V$  be a maximal strongly  $H$ -independent subset of the additive group  $V$ . Since for each subset  $B$  of  $V$  the group hull  $H(B)$  of  $B$  coincides with its linear hull, the operator of group (=linear) hull in  $V$  is of matroid type. By Proposition 1.1, the maximal strongly  $H$ -independent subset  $M$  of  $V$  is maximal  $H$ -independent in  $V$ .

We need to check that  $M$  remains maximal  $H$ -independent in the group  $G$ . Given any element  $g \in G \setminus M$  we need to show that the union  $M \cup \{g\}$  is not  $H$ -independent in  $G$ . If  $g \in V$ , then the set  $M \cup \{g\} \subset V$  is not  $H$ -independent in  $G$  by the maximality of the  $H$ -independent set  $M$  in  $V$ .

So, we assume that  $g \notin V$ . In this case  $g = (u, f) \in V \times GL(V)$  for some non-identity automorphism  $f$  of  $V$ . Observe that  $g^{-1} = (-f^{-1}(v), f^{-1})$  and for each  $v \in V$  we get

$$\begin{aligned} gvg^{-1} &= (u, f) \cdot (v, \text{id}_V) \cdot (-f^{-1}(u), f^{-1}) = (f(v), \text{id}_V) = f(v) \text{ and} \\ g^{-1}vg &= (-f^{-1}(u), f^{-1}) \cdot (v, \text{id}_V) \cdot (u, f) = (f^{-1}(v), \text{id}_V) = f^{-1}(v). \end{aligned}$$

Being a maximal linearly independent subset of  $V$ , the set  $M$  is a Hamel basis in  $V$ . Since  $f \neq \text{id}_V$ , there is a point  $a \in M$  with  $f(a) \neq a$ . Since  $M$  is a Hamel basis of the linear space  $V$ , each point  $v \in V$  can be written as the sum  $\Sigma B = \sum_{b \in B} b$  of a unique finite subset  $B \subset M$ . In particular,  $a = \Sigma A$  and  $f(a) = \Sigma F$  for some finite subsets  $A, F \subset M$ . Since  $F \neq A$ , there is a point  $v \in (A \setminus F) \cup (F \setminus A)$ .

If  $v \in A \setminus F$ , then consider the set  $A_v = A \setminus \{v\} \subset M \setminus \{v\}$  and observe that

$$v = a - \Sigma A_v = f^{-1}(f(a)) - \Sigma A_v = g^{-1} \cdot (\Sigma F, \text{id}_A) \cdot g \cdot (\Sigma A_v, \text{id}_A)^{-1} \in H(\{g\} \cup M \setminus \{v\}),$$

which implies that  $M \cup \{g\}$  is not  $H$ -independent in  $G$ .

If  $v \in F \setminus A$ , then consider the set  $F_v = F \setminus \{v\} \subset M \setminus \{v\}$  and observe that

$$v = \Sigma F - \Sigma F_v = f(a) - \Sigma F_v = g \cdot (\Sigma A, \text{id}_V) \cdot g^{-1} \cdot (\Sigma F_v, \text{id}_V)^{-1} \in H(\{g\} \cup M \setminus \{v\}),$$

which implies that  $M \cup \{g\}$  is not  $H$ -independent in  $G$ .  $\square$

So, in general, hull operators generated by algebraic structures need not be of matroid type, which makes the problem of constructing large (strongly) independent sets non-trivial. We shall be interested in hull operators induced by the structure of universal algebra (which includes as partial cases the structures of group, linear space, linear algebra, etc.) Such hull operators will be defined and studied in the next section.

## 2. UNIVERSAL ALGEBRAS

A *universal algebra* is a pair  $\mathbb{A} = (A, \mathcal{A})$  consisting of a set  $A$  and a family  $\mathcal{A}$  of algebraic operations on  $A$ . An *algebraic operation* on a set  $A$  is any function  $\alpha : A^{S_\alpha} \rightarrow A$  defined on a finite power  $A^{S_\alpha}$  of  $A$ , where  $S_\alpha$  is a finite subset of  $\omega$  called *the support* of the operation  $\alpha$ . An algebraic operation  $\alpha : A^{S_\alpha} \rightarrow A$  is *constant* if  $\alpha(A^{S_\alpha})$  is a singleton. In particular, each algebraic operation  $\alpha : A^\emptyset \rightarrow A$  with empty support is constant.

Observe that any function  $\sigma : F \rightarrow E$  between finite subsets of  $\omega$  induces a dual function  $\sigma^* : A^E \rightarrow A^F$ ,  $\sigma^* : f \mapsto f \circ \sigma$ , called a *substitution operator*. Then for any algebraic operation  $\alpha : A^{S_\alpha} \rightarrow A$  with support  $S_\alpha = F$  the composition  $\beta = \alpha \circ \sigma^*$  is a well-defined algebraic operation on  $A$  with support  $S_\beta = E$ .

A family  $\mathcal{A}$  of operations on a set  $A$  is called

- *unital* if  $\mathcal{A}$  contains the identity operation  $\text{id}_A : A^1 \rightarrow A$ ,  $\text{id}_A : (a_0) \mapsto a_0$ ;
- $\emptyset$ -*regular* if for each constant operation  $\alpha \in \mathcal{A}$  there is an operation  $\beta \in \mathcal{A}$  with empty support  $S_\beta = \emptyset$  such that  $\beta(A^\emptyset) = \alpha(A^{S_\alpha})$ ;
- *stable under substitutions* (briefly, *substitution-stable*) if for any function  $\sigma : F \rightarrow E$  between finite subsets of  $\omega$  and any algebraic operation  $\alpha \in \mathcal{A}$  with  $\text{supp}(\alpha) = F$  the algebraic operation  $\alpha \circ \sigma^* : A^E \rightarrow A$  belongs to  $\mathcal{A}$ ;
- *stable under compositions* if for any finite subset  $S \subset \omega$  and algebraic operations  $\alpha \in \mathcal{A}$  and  $\alpha_i \in \mathcal{A}$ ,  $i \in S_\alpha$ , with supports  $\text{supp}(\alpha_i) = S$  for all  $i \in E$  the composition  $\alpha \circ (\alpha_i)_{i \in S_\alpha} : A^S \rightarrow A$  of the diagonal product  $(\alpha_i)_{i \in S_\alpha} : A^S \rightarrow A^{S_\alpha}$  with  $\alpha$  belongs to  $\mathcal{A}$ ;
- a *clone* if  $\mathcal{A}$  is unital,  $\emptyset$ -regular, and stable under substitutions and compositions.

The *clone* of a universal algebra  $\mathbb{A} = (A, \mathcal{A})$  is the universal algebra  $\bar{\mathbb{A}} = (A, \bar{\mathcal{A}})$  endowed with the smallest clone  $\bar{\mathcal{A}}$  that contains the operation family  $\mathcal{A}$ . The clone  $\bar{\mathcal{A}}$  is equal to the union  $\bar{\mathcal{A}} = \bigcup_{n \in \omega} \mathcal{A}_n$  of operation families  $\mathcal{A}_n$ ,  $n \in \omega$ , defined by induction. Let  $\mathcal{A}_0 = \{\text{id}_A\}$  and

$$\begin{aligned} \mathcal{A}_{n+1} &= \mathcal{A}_n \cup \{\alpha \in A^{A^\emptyset} : \exists \beta \in \mathcal{A}_n \ \alpha(A^\emptyset) = \beta(A^{S_\beta})\} \cup \\ &\cup \{\alpha \circ \sigma^* : \alpha \in \mathcal{A}_n, \ \sigma : S_\alpha \rightarrow F \text{ is a function into a finite subset } F \subset \omega\} \cup \\ &\cup \{\alpha \circ (\alpha_i)_{i \in S_\alpha} : \alpha \in \mathcal{A}, \ (\alpha_i)_{i \in S_\alpha} \in (\mathcal{A}_n)^{S_\alpha}, \ \forall i, j \in S_\alpha \ S_{\alpha_i} = S_{\alpha_j}\} \end{aligned}$$

for  $n \in \omega$ . This implies that the clone  $\bar{\mathcal{A}}$  of  $\mathcal{A}$  has cardinality  $|\bar{\mathcal{A}}| \leq \max\{|\mathcal{A}|, \aleph_0\}$ .

Each universal algebra  $\mathbb{A} = (A, \mathcal{A})$  possesses the *canonical hull operator*  $\mathcal{A}(\cdot) : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  assigning to each subset  $B \subset A$  its  $\mathcal{A}$ -*hull*

$$\mathcal{A}(B) = B \cup \bigcup_{\alpha \in \mathcal{A}} \alpha(B^{S_\alpha}).$$

The definition implies that this hull operation has finite supports.

If the operation family  $\mathcal{A}$  is  $\emptyset$ -regular, then for each constant algebraic operation  $\alpha \in \mathcal{A}$ , the singleton  $\alpha(A^{n_\alpha})$  lies in the  $\mathcal{A}$ -hull  $\mathcal{A}(\emptyset)$  of the empty subset of  $A$ .

**Proposition 2.1.** *Let  $\mathbb{A} = (A, \mathcal{A})$  be a universal algebra whose operation family  $\mathcal{A}$  is unital and stable under substitutions. Then for each subset  $B \subset A$  and a point  $a \in \mathcal{A}(B)$  there is an algebraic operation  $\alpha \in \mathcal{A}$  and an injective function  $x : S_\alpha \rightarrow B$  such that  $a = \alpha(x)$ .*

*Proof.* Since  $\mathcal{A}$  is unital,  $\mathcal{A}(B) = \bigcup_{\beta \in \mathcal{A}} \alpha(B^{S_\beta})$  and we can find an operation  $\beta \in \mathcal{A}$  and a function  $z : S_\beta \rightarrow B$  such that  $a = \beta(z)$ . Let  $x : S \rightarrow z(S_\beta) \subset B$  be any bijective map defined on a finite subset  $S \subset \omega$ . Consider the function  $\sigma = x^{-1} \circ z : S_\beta \rightarrow S$ , which induces the substitution operator  $\sigma^* : A^S \rightarrow A^{S_\beta}$ . Since  $\mathcal{A}$  is stable under substitutions, the operation  $\alpha = \beta \circ \sigma^* : A^S \rightarrow A$  belongs to  $\mathcal{A}$  and has support  $S_\alpha = S$ . Moreover,  $\alpha(x) = \beta \circ \sigma^*(x) = \beta(x \circ \sigma) = \beta(z) = a$ .  $\square$

A subset  $B \subset A$  is called a *subalgebra* of a universal algebra  $\mathbb{A} = (A, \mathcal{A})$  if  $\mathcal{A}(B) \subset B$ , i.e.,  $B$  coincides with its  $\mathcal{A}$ -hull  $\mathcal{A}(B)$ . Since the intersection of subalgebras is a subalgebra, for each subset  $B \subset A$  there is the smallest subalgebra of  $\mathbb{A}$  that contains  $B$ . This subalgebra is called the *subalgebra generated by  $B$*  and admits the following simple description:

**Proposition 2.2.** *Let  $\mathbb{A} = (A, \mathcal{A})$  be a universal algebra and  $\bar{\mathbb{A}} = (\bar{A}, \bar{\mathcal{A}})$  be its clone. For each subset  $B \subset A$  the subalgebra generated by  $B$  coincides with the  $\bar{\mathcal{A}}$ -hull  $\bar{\mathcal{A}}(B)$  of  $B$ .*

*Proof.* Let  $\langle B \rangle$  denote the subalgebra of  $\mathbb{A}$  generated by  $B$ . The inclusion  $\langle B \rangle \subset \bar{\mathcal{A}}(B)$  will follow as soon as we check that the  $\bar{\mathcal{A}}$ -hull  $\bar{\mathcal{A}}(B)$  of  $B$  is a subalgebra of the universal algebra  $\mathbb{A}$ , that is  $\bar{\mathcal{A}}(B)$  contains all constants and it is closed all operations from  $\mathcal{A}$ . We need to check that  $\mathcal{A}(\bar{\mathcal{A}}(B)) \subset \bar{\mathcal{A}}(B)$ . Take any element  $y \in \mathcal{A}(\bar{\mathcal{A}}(B))$  and find an operation  $\alpha \in \mathcal{A}$  and a function  $x : S_\alpha \rightarrow \bar{\mathcal{A}}(B)$  such that  $y = \alpha(x)$ . For every  $i \in S_\alpha$  the point  $x(i)$  belongs to the  $\bar{\mathcal{A}}$ -hull  $\bar{\mathcal{A}}(B)$  of  $B$  and hence can be written as  $x(i) = \alpha_i(z_i)$  for some algebraic operation  $\alpha_i \in \bar{\mathcal{A}}$  and some function  $z_i : S_{\alpha_i} \rightarrow B$ . Choose a finite subset  $S \subset \omega$  of cardinality  $|S| = \sum_{i \in S_\alpha} |S_{\alpha_i}|$  and for every  $i \in S_\alpha$  choose an injective function  $\sigma_i : S_{\alpha_i} \rightarrow S$  so that  $\sigma_i(S_{\alpha_i}) \cap \sigma_j(S_{\alpha_j}) = \emptyset$  for  $i \neq j$ . Each function  $\sigma_i$  induces the surjective substitution operators  $\sigma_i^* : A^S \rightarrow A^{S_{\alpha_i}}$ ,  $\sigma_i^* : f \mapsto f \circ \sigma_i$ . Consider a unique function  $z : S \rightarrow B$  such that  $z \circ \sigma_i = z_i$  for all  $i \in S_\alpha$ .

Since  $\bar{\mathcal{A}}$  is a clone, it is closed under substitutions. Consequently, for every  $i \in S_\alpha$  the operation  $\beta_i = \alpha_i \circ \sigma_i^* : A^S \rightarrow A$  belongs to  $\bar{\mathcal{A}}$ . It follows that  $\beta_i(z) = \alpha_i \circ \sigma_i^*(z) = \alpha(z \circ \sigma_i) = \alpha_i(z_i) = x(i)$ . Since the function family  $\bar{\mathcal{A}}$  is closed under compositions, the operation  $\beta = \alpha \circ (\beta_i)_{i \in S_\alpha} : A^S \rightarrow A$  belongs to  $\bar{\mathcal{A}}$ . Since

$$\beta(z) = \alpha((\beta_i(z))_{i \in S_\alpha}) = \alpha((x(i))_{i \in S_\alpha}) = \alpha(x) = y,$$

the point  $y = \beta(z)$  belongs to  $\bar{\mathcal{A}}(B)$ . So,  $\mathcal{A}(\bar{\mathcal{A}}(B)) \subset \bar{\mathcal{A}}(B)$  and  $\bar{\mathcal{A}}(B)$  is a subalgebra of  $\mathbb{A}$ , which implies  $\langle B \rangle \subset \bar{\mathcal{A}}(B)$ .

To prove that  $\bar{\mathcal{A}}(B) \subset \langle B \rangle$ , we use the decomposition  $\bar{\mathcal{A}} = \bigcup_{n \in \omega} \mathcal{A}_n$  of  $\bar{\mathcal{A}}$  into the countable union of the operation families  $\mathcal{A}_n$ ,  $n \in \omega$ , defined at the beginning of Section 2 right after the definition of clone. Since  $\bar{\mathcal{A}}(B) = \bigcup_{n \in \omega} \mathcal{A}_n(B)$ , it suffices to check that  $\mathcal{A}_n(B) \subset \langle B \rangle$  for every  $n \in \omega$ . This will be done by induction on  $n \in \omega$ .

Since  $\mathcal{A}_0 = \{\text{id}_A\}$ ,  $\mathcal{A}_0(B) = B \subset \langle B \rangle$ . Assume that for some  $n \in \omega$  we have proved that  $\mathcal{A}_n(B) \subset \langle B \rangle$ . The inclusion  $\mathcal{A}_{n+1}(B) \subset \langle B \rangle$  will follow as soon as we check that  $\beta(x) \in \langle B \rangle$  for each operation  $\beta \in \mathcal{A}_{n+1}$  and a function  $x : S_\beta \rightarrow B$ . If  $\beta \in \mathcal{A}_n$ , then  $\beta(f) \in \mathcal{A}_n(B) \subset \langle B \rangle$ .

If  $\beta \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n$ , then by the definition of the operation family  $\mathcal{A}_{n+1}$ , the following three cases are possible:

1)  $S_\beta = \emptyset$  and there is a constant operation  $\alpha \in \mathcal{A}_n$  such that  $\beta(A^\emptyset) = \alpha(A^{S_\alpha})$ . If  $B = \emptyset$ , then  $x \in B^{S_\alpha}$  implies  $S_\alpha = \emptyset = S_\beta$  and hence  $\beta(x) = \alpha(x) \in \mathcal{A}_n(B) \subset \langle B \rangle$ .

If  $B \neq \emptyset$ , then we can take any function  $y : S_\alpha \rightarrow B$  and conclude that  $\beta(x) = \alpha(y) \in \mathcal{A}_n(B) \subset \langle B \rangle$ .

2)  $\beta = \alpha \circ \sigma^*$  for some operation  $\alpha \in \mathcal{A}_n$  and some function  $\sigma : S_\beta \rightarrow S_\alpha$ . Consider the function  $y = x \circ \sigma : S_\beta \rightarrow B$  and observe that  $\beta(x) = \alpha \circ \sigma^*(x) = \alpha(x \circ \sigma) \in \mathcal{A}_n(B) \subset \langle B \rangle$ .

3)  $\beta = \alpha \circ (\alpha_i)_{i \in S_\alpha}$  for some operations  $\alpha \in \mathcal{A}$  and  $(\alpha_i)_{i \in S_\alpha} \in \mathcal{A}_n^{S_\alpha}$  with  $S_{\alpha_i} = S_\beta$  for all  $i \in S_\alpha$ . Since  $x \in B^{S_\beta}$ , the inductive assumption  $\mathcal{A}_n(B) \subset \langle B \rangle$  guarantees that for every  $i \in S_\alpha$  the point  $y(i) = \alpha_i(x) \in \mathcal{A}_n(B)$  belongs to  $\langle B \rangle$ . Consider the function  $y : S_\alpha \rightarrow \langle B \rangle$ ,  $y : i \mapsto y(i) = \alpha_i(x)$ . Then  $\beta(x) = \alpha((\alpha_i(x))_{i \in S_\alpha}) = \alpha((y(i))_{i \in S_\alpha}) = \alpha(y) \in \mathcal{A}(\langle B \rangle) \subset \langle B \rangle$ . The last inclusion  $\mathcal{A}(\langle B \rangle) \subset \langle B \rangle$  follows from the fact that  $\langle B \rangle$  is a subalgebra of  $\mathbb{A}$ .  $\square$

Let  $\mathbb{A} = (A, \mathcal{A})$  be a universal algebra,  $X$  be a non-empty set, and  $A^X$  be the set of all functions from  $X$  to  $A$ . For every  $x \in X$  denote by  $\delta_x : A^X \rightarrow A$ ,  $\delta_x : f \mapsto f(x)$ , the projection onto  $x$ -th coordinate.

Observe that each algebraic operation  $\alpha : A^{S_\alpha} \rightarrow A$  induces an algebraic operation  $\alpha^X : (A^X)^{S_\alpha} \rightarrow A^X$  on the set  $A^X$  of all functions from  $X$  to  $A$ . The operation  $\alpha^X$  assigns to each function  $f : S_\alpha \rightarrow A^X$  the function  $\alpha^X(f) : X \rightarrow A$  defined by  $\alpha^X(f) : x \mapsto \alpha(\delta_x \circ f)$  for  $x \in X$ . Writing the function  $f$  in coordinates as  $f = (f_i)_{i \in S_\alpha}$ , we get that  $\alpha^X(f) = \alpha^X((f_i)_{i \in S_\alpha})$  is the function assigning to each  $x \in X$  the point  $\alpha((f_i(x))_{i \in S_\alpha})$ .

For a universal algebra  $\mathbb{A} = (A, \mathcal{A})$  its  $X$ -th power is the pair  $\mathbb{A}^X = (A^X, \mathcal{A}^X)$  consisting of the  $X$ -th power of  $A$  and the operation family  $\mathcal{A}^X = \{\alpha^X\}_{\alpha \in \mathcal{A}}$ .

Now let us consider an important example of a universal algebra  $\mathbf{2} = (2, \mathcal{B})$ , called the *Boolean clone*. It consists of the doubleton  $2 = \{0, 1\}$  and the family  $\mathcal{B}$  of all possible algebraic operations on 2. In the next section, we shall see that the powers  $\mathbf{2}^X$  of the Boolean clone play an important role in studying independent and free sets in powers of arbitrary universal algebras.

### 3. INDEPENDENT AND FREE SETS IN UNIVERSAL ALGEBRAS

In this section we introduce and study three independence notions in universal algebras.

A subset  $B \subset A$  of a universal algebra  $\mathbb{A} = (A, \mathcal{A})$  is called

- $\mathcal{A}$ -independent if  $b \notin \mathcal{A}(B \setminus \{b\})$  for all  $b \in B$ ;
- strongly  $\mathcal{A}$ -independent if  $B \cap \mathcal{A}(\emptyset) = \emptyset$  and  $\mathcal{A}(B_1) \cap \mathcal{A}(B_2) = \mathcal{A}(B_1 \cap B_2)$  for any subsets  $B_1, B_2 \subset B$ ;
- $\mathcal{A}$ -free if for any function  $f : B \rightarrow A$  there is a function  $\tilde{f} : \mathcal{A}(B) \rightarrow A$  such that  $\tilde{f} \circ \alpha(x) = \alpha(\tilde{f} \circ x)$  for any algebraic operation  $\alpha \in \mathcal{A}$  and any function  $x \in B^{S_\alpha} \subset A^{S_\alpha}$ .

The notion of a (strongly)  $\mathcal{A}$ -independent set is induced by the operator of  $\mathcal{A}$ -hull. On the other hand, the notion of an  $\mathcal{A}$ -free set is specific for universal algebras and has no hull counterpart.

The definitions imply that the notions of  $\mathcal{A}$ -independent and  $\mathcal{A}$ -free sets are “monotone” which respect to  $\mathcal{A}$ :

**Proposition 3.1.** *If a subset  $B \subset A$  of a universal algebra  $(A, \mathcal{A})$  is  $\mathcal{A}$ -independent ( $\mathcal{A}$ -free), then it is  $\mathcal{A}'$ -independent ( $\mathcal{A}'$ -free) for each operation family  $\mathcal{A}' \subset \mathcal{A}$ .*

On the other hand, the strong  $\mathcal{A}$ -independence is not monotone with respect to  $\mathcal{A}$ .

**Example 3.2.** Take a set  $A$  of cardinality  $|A| \geq 2$  and consider an operation family  $\mathcal{A} = \{\alpha, \beta\}$  consisting of two constant operations  $\alpha : A^1 \rightarrow A$ ,  $\beta : A^0 \rightarrow A$  with  $\alpha(A^1) = \beta(A^0)$ . Observe that each subset of  $A$  is strongly  $\mathcal{A}$ -independent while each subset  $B \subset A$  of cardinality  $|B| \geq 2$  fails to be strongly  $\mathcal{A}'$ -independent for the subfamily  $\mathcal{A}' = \{\alpha\}$ . Indeed, take two non-empty disjoint subsets  $B_1, B_2 \subset B$  and observe that  $\mathcal{A}'(B_1) \cap \mathcal{A}'(B_2) = \{\alpha(A^1)\} \neq \emptyset = \mathcal{A}'(\emptyset)$ .

Now we shall characterize  $\mathcal{A}$ -free sets in universal algebras. On first characterization follows immediately from the definition.

**Proposition 3.3.** *A subset  $B \subset A$  of a universal algebra  $(A, \mathcal{A})$  is  $\mathcal{A}$ -free if and only if for any algebraic operations  $\alpha, \beta \in \mathcal{A}$  and functions  $x \in B^{S_\alpha}$ ,  $y \in B^{S_\beta}$  the equality  $\alpha(x) = \beta(y)$  implies that  $\alpha(f \circ x) = \beta(f \circ y)$  for any function  $f : B \rightarrow A$ .*

For unital substitution-stable universal algebras this characterization can be improved as follows.

**Proposition 3.4.** *A subset  $B \subset A$  of a unital substitution-stable universal algebra  $(A, \mathcal{A})$  is  $\mathcal{A}$ -free if and only if for any distinct algebraic operations  $\alpha, \beta \in \mathcal{A}$  with  $S_\alpha = S_\beta$  the inequality  $\alpha(x) \neq \beta(x)$  holds for each injective function  $x \in B^{S_\alpha} = B^{S_\beta}$ .*

*Proof.* To prove the “only if” part, assume that the set  $B$  is  $\mathcal{A}$ -free. Fix two distinct algebraic operations  $\alpha, \beta \in \mathcal{A}$  such that  $S_\alpha = S_\beta = S$  for some finite set  $S \subset \omega$ . We need to show that  $\alpha(x) \neq \beta(x)$  for each injective function  $x \in B^S$ .

Since  $\alpha \neq \beta$ , there is a function  $y \in B^S$  such that  $\alpha(y) \neq \beta(y)$ . Using the injectivity of  $x$ , choose a function  $f : B \rightarrow A$  such that  $f|_{x(S)} = y \circ x^{-1}|_{x(S)}$ . Since  $B$  is  $\mathcal{A}$ -free, there is a function  $\tilde{f} : \mathcal{A}(B) \rightarrow A$  such that  $\tilde{f}(\gamma(z)) = \gamma(\tilde{f} \circ z)$  for any  $\gamma \in \mathcal{A}$  and  $z \in A^{S_\gamma}$ . In particular,  $\tilde{f}(\alpha(x)) = \alpha(\tilde{f} \circ x) = \alpha(y) \neq \beta(y) = \beta(\tilde{f} \circ x) = \tilde{f}(\beta(x))$ , which implies  $\alpha(x) \neq \beta(x)$ .

To prove the “if” part, assume that the set  $B$  is not  $\mathcal{A}$ -free. Applying Proposition 3.3, find algebraic operations  $\alpha, \beta \in \mathcal{A}$  and functions  $x \in B^{S_\alpha}$ ,  $y \in B^{S_\beta}$  such that  $\alpha(x) = \beta(y)$  and  $\alpha(f \circ x) \neq \beta(f \circ y)$  for some function  $f : B \rightarrow A$ . Fix any bijective function  $z : S \rightarrow x(S_\alpha) \cup y(S_\beta) \subset B$  defined on a finite subset  $S \subset \omega$ . Consider the functions  $\sigma_\alpha = z^{-1} \circ x : S_\alpha \rightarrow S$  and  $\sigma_\beta = z^{-1} \circ y : S_\beta \rightarrow S$ , which induce the substitution operators  $\sigma_\alpha^* : A^S \rightarrow A^{S_\alpha}$  and  $\sigma_\beta^* : A^S \rightarrow A^{S_\beta}$ . Since the operation family  $\mathcal{A}$  is stable under substitutions, the algebraic operations  $\tilde{\alpha} = \alpha \circ \sigma_\alpha^* : A^S \rightarrow A$  and  $\tilde{\beta} = \beta \circ \sigma_\beta^* : A^S \rightarrow A$  belong to the family  $\mathcal{A}$ . Observe that  $\tilde{\alpha}(z) = \alpha \circ \sigma_\alpha^*(z) = \alpha(z \circ \sigma_\alpha) = \alpha(x) = \beta(y) = \beta(z \circ \sigma_\beta) = \tilde{\beta}(z)$ . On the other hand,

$$\tilde{\alpha}(f \circ z) = \alpha \circ \sigma_\alpha^*(f \circ z) = \alpha(f \circ z \circ \sigma_\alpha) = \alpha(f \circ x) \neq \beta(f \circ y) = \beta(f \circ z \circ \sigma_\beta) = \tilde{\beta}(f \circ z)$$

implies that  $\tilde{\alpha} \neq \tilde{\beta}$ . □

By Proposition 1.1, each strongly  $\mathcal{A}$ -independent subset  $B \subset A$  of a universal algebra  $(A, \mathcal{A})$  is  $\mathcal{A}$ -independent.

**Proposition 3.5.** *Let  $(A, \mathcal{A})$  be a unital  $\emptyset$ -regular substitution-stable universal algebra of cardinality  $|A| \geq 2$ . Each  $\mathcal{A}$ -free subset  $B \subset A$  is strongly  $\mathcal{A}$ -independent.*



*Proof.* First we prove that  $B \cap \mathcal{A}(\emptyset) = \emptyset$ . Assume conversely that  $B \cap \mathcal{A}(\emptyset)$  contains some point  $b$ .

Since  $|A| \geq 2$ , we can find two functions  $f_1, f_2 : B \rightarrow A$  such that  $f_1(b) \neq f_2(b)$ . Since  $B$  is  $\mathcal{A}$ -free, for every  $i \in \{1, 2\}$ , there is a function  $\bar{f}_i : \mathcal{A}(B) \rightarrow A$  such that  $\bar{f}_i \circ \alpha(x) = \alpha(f_i \circ x)$  for any  $\alpha \in \mathcal{A}$  and  $x \in B^{S_\alpha}$ .

The universal algebra  $(A, \mathcal{A})$  is unital and hence contains the identity algebraic operation  $\alpha : A^1 \rightarrow A$ ,  $\alpha : (a) \mapsto a$ . For this operation we get  $\bar{f}_i(b) = \bar{f}_i \circ \alpha(b) = \alpha(f_i(b)) = f_i(b)$  and hence  $\bar{f}_1(b) \neq \bar{f}_2(b)$ .

On the other hand, the inclusion  $b \in \mathcal{A}(\emptyset)$  yields a 0-ary operation  $\beta \in \mathcal{A}$  such that  $b = \beta(\emptyset)$  where  $\emptyset : \emptyset \rightarrow B$  is the unique element of the power  $B^\emptyset = B^{S_\beta}$ . Then for every  $i \in \{1, 2\}$ , the choice of  $\bar{f}_i$  guarantees that  $\bar{f}_i(b) = \bar{f}_i \circ \beta(\emptyset) = \beta(f_i \circ \emptyset) = \beta(\emptyset) = b$  and hence  $\bar{f}_1(b) = b = \bar{f}_2(b)$ , which contradicts the inequality  $\bar{f}_1(b) \neq \bar{f}_2(b)$  proved earlier. Hence  $B \cap \mathcal{A}(\emptyset) = \emptyset$ .

Next, we prove that  $\mathcal{A}(B_1) \cap \mathcal{A}(B_2) = \mathcal{A}(B_1 \cap B_2)$  for any subsets  $B_1, B_2 \subset B$ . This equality is trivial if  $B_1 \subset B_2$  or  $B_2 \subset B_1$ . So we assume that both complements  $B_1 \setminus B_2$  and  $B_2 \setminus B_1$  are not empty. Assume that  $\mathcal{A}(B_1) \cap \mathcal{A}(B_2) \neq \mathcal{A}(B_1 \cap B_2)$  and find a point  $a \in \mathcal{A}(B_1) \cap \mathcal{A}(B_2) \setminus \mathcal{A}(B_1 \cap B_2)$ . For the point  $a \in \mathcal{A}(B_1) \cap \mathcal{A}(B_2)$ , there are algebraic operations  $\alpha, \beta \in \mathcal{A}$  and functions  $x \in B_1^{S_\alpha}$ ,  $y \in B_2^{S_\beta}$  such that  $\alpha(x) = a = \beta(y)$ . Using a similar reasoning as in the proof of Proposition 3.4 we may assume that  $x$  and  $y$  are injective

Let us show that the operation  $\alpha$  is not constant. In the opposite case, the  $\emptyset$ -regularity of  $\mathcal{A}$ , yields a 0-ary operation  $\gamma \in \mathcal{A}$  such that  $\alpha(A^{S_\alpha}) = \gamma(A^\emptyset)$ . Then  $a = \alpha(x) = \gamma(\emptyset) \in \mathcal{A}(\emptyset) \subset \mathcal{A}(B_1 \cap B_2)$ , which contradicts the choice of  $a$ .

Next, we prove that the intersection  $x(S_\alpha) \cap y(S_\beta) \subset B_1 \cap B_2$  is not empty. Assuming the converse and using the fact that the operation  $\alpha$  is not constant, find a function  $x' \in A^{S_\alpha}$  such that  $\alpha(x') \neq \beta(y)$ . Since the subsets  $x(S_\alpha)$  and  $y(S_\beta)$  of  $B$  are disjoint, we can find a function  $f : B \rightarrow A$  such that  $f|_{x(S_\alpha)} = x' \circ x^{-1}$  and  $f|_{y(S_\beta)} = y \circ y^{-1}$ . Then

$$\bar{f}(a) = \bar{f}(\alpha(x)) = \alpha(f \circ x) = \alpha(x') \neq \beta(y) = \beta(f \circ y) = \bar{f}(\beta(y)) = \bar{f}(a),$$

which is a contradiction.

Thus the intersection  $x(S_\alpha) \cap y(S_\beta) \subset B_1 \cap B_2$  is not empty and we can choose a function  $f : B \rightarrow A$  such that  $f|_{y(S_\beta)} = \text{id}_{y(S_\beta)}$  and  $f(x(S_\alpha) \setminus y(S_\beta)) \subset B_1 \cap B_2$ . Such a choice of  $f$  guarantees that  $f \circ x \in B_1 \cap B_2$  and then  $\alpha(f \circ x) \in \mathcal{A}(B_1 \cap B_2) \not\equiv a$  implies that  $a \neq \alpha(f \circ x)$ . Since the set  $B$  is  $\mathcal{A}$ -free, the function  $f$  can be extended to a function  $\bar{f} : \mathcal{A}(B) \rightarrow A$  such that  $\bar{f}(\gamma(z)) = f \circ \gamma(z)$  for each  $\gamma \in \mathcal{A}$  and  $z \in B^{S_\gamma}$ . In particular,

$$\bar{f}(a) = \bar{f}(\beta(y)) = \beta(f \circ y) = \beta(y) = a \neq \alpha(f \circ x) = \bar{f}(\alpha(x)) = \bar{f}(a).$$

This contradiction completes the proof of the strong  $\mathcal{A}$ -independence of the set  $B$ .  $\square$

Proposition 3.5 shows that for a subset of a unital  $\emptyset$ -regular substitution-stable universal algebra  $(A, \mathcal{A})$  we have the following implications:

$$\mathcal{A}\text{-free} \Rightarrow \text{strongly } \mathcal{A}\text{-independent} \Rightarrow \mathcal{A}\text{-independent}.$$

The first implication cannot be reversed as shown by the following simple example.

**Example 3.6.** There is a clone  $(A, \bar{\mathcal{A}})$  containing an infinite strongly  $\bar{\mathcal{A}}$ -independent subset  $B \subset X$  such that each  $\bar{\mathcal{A}}$ -free subset of  $A$  is empty.

*Proof.* Consider the linear algebra  $c_{00}$  consisting of all functions  $x : \omega \rightarrow \mathbb{R}$  with finite support  $\text{supp}(x) = \{n \in \omega : x(n) \neq 0\} \subset \omega$ . This algebra is endowed with the operation family  $\mathcal{A} = \{+, \cdot\} \cup \{\alpha_t : t \in \mathbb{R}\}$  consisting of two binary operations (of addition and multiplication) and continuum many unary operations  $\alpha_t : \bar{x} \mapsto t \cdot \bar{x}$  of multiplication by a real number  $t$ . The clone  $\bar{\mathcal{A}}$  of  $\mathcal{A}$  contains the subfamily  $\mathcal{A}'$  consisting of all polynomials  $p(x) = \sum_{i=0}^n \lambda_i x^{i+1}$  of one variable, equal to zero at the zero function.

**Claim 3.7.** The set  $B = \{x \in c_{00} : x(\omega) \subset \{0, 1\}, |\text{supp}(x)| = 1\}$  of characteristic functions of singletons is strongly  $\bar{\mathcal{A}}$ -independent.

*Proof.* It is easy to check that for each subset  $B' \subset B$  its  $\bar{\mathcal{A}}$ -hull  $\bar{\mathcal{A}}(B')$  is equal to the linear subspace  $\mathbb{R}^{\text{supp}(B')} = \{x \in c_{00} : \text{supp}(x) \subset \text{supp}(B')\}$  where  $\text{supp}(B') = \bigcup_{b \in B'} \text{supp}(b)$ . It follows that  $B \cap \bar{\mathcal{A}}(\emptyset) = B \cap \{0\} = \emptyset$  and for any two subsets  $B_1, B_2 \subset B$  we get

$$\bar{\mathcal{A}}(B_1) \cap \bar{\mathcal{A}}(B_2) = \mathbb{R}^{\text{supp}(B_1)} \cap \mathbb{R}^{\text{supp}(B_2)} = \mathbb{R}^{\text{supp}(B_1) \cap \text{supp}(B_2)} = \mathbb{R}^{\text{supp}(B_1 \cap B_2)} = \bar{\mathcal{A}}(B_1 \cap B_2),$$

which means that the subset  $B$  is strongly independent.  $\square$

**Claim 3.8.** Any non-empty subset  $B \subset A$  is not  $\mathcal{A}'$ -free and hence is not  $\bar{\mathcal{A}}$ -free.

*Proof.* Fix any function  $b \in B$ . This function has finite support  $F = \text{supp}(b)$ . Then  $b$  and all its finite powers  $b^n$ ,  $n > 0$ , belong to the finite-dimensional linear subspace  $\mathbb{R}^F = \{x \in c_{00} : \text{supp}(x) \subset F\}$  of  $c_{00}$ . Consequently, the set  $\{b^{n+1} : 0 \leq n \leq |F|\}$  is linearly dependent, which allows us to find a non-zero vector  $(\lambda_0, \dots, \lambda_{|F|}) \in \mathbb{R}^{|F|+1}$  such that  $\sum_{i=0}^{|F|} \lambda_i b^{i+1} = 0$ . This means that  $p(b) = 0$  for the non-zero polynomial  $p(x) = \sum_{i=0}^{|F|} \lambda_i x^{i+1}$ . Since the

polynomial  $p \in \mathcal{A}'$  is non-zero, there is a vector  $y \in \mathbb{R}^F$  such that  $p(y) \neq 0$ . Let  $f : B \rightarrow c_{00}$  be any function such that  $f(b) = y$ .

Assuming that the set  $B$  is  $\mathcal{A}'$ -free, we could find a function  $\bar{f} : \mathcal{A}'(B) \rightarrow c_{00}$  such that  $\bar{f}(p(b)) = p(f \circ b)$ . But  $\bar{f}(p(b)) = 0 \neq p(y) = p(f(b))$ . This contradiction completes the proof.  $\square$

$\square$

#### 4. FREE SETS IN POWERS OF UNIVERSAL ALGEBRAS

In this section we shall construct free sets of large cardinality in powers of universal algebras. We start with studying free sets in a power  $2^X = (2^X, \mathcal{B}^X)$  of the Boolean clone  $2 = (2, \mathcal{B})$  consisting of the doubleton  $2 = \{0, 1\}$  and the family  $\mathcal{B}$  of all possible algebraic operations on 2.

It turns out that  $\mathcal{B}^X$ -free subsets of  $2^X$  coincide with independent sets, well studied in Set Theory [6, §17].

Let us recall [6, p.83] that a family  $\mathcal{F}$  of subsets of a set  $X$  is called *independent* if for any finite disjoint sets  $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$  the intersection

$$\bigcap_{F \in \mathcal{F}_1} F \cap \bigcap_{F \in \mathcal{F}_2} (X \setminus F)$$

is not empty.

Identifying each subset  $F \subset X$  with its characteristic function  $\chi_F : X \rightarrow 2 = \{0, 1\}$ , we can reformulate the notion of an independent family of sets in the language of an independent function family. Namely, a function family  $\mathcal{F} \subset 2^X$  is *independent* if for any pairwise distinct functions  $f_0, \dots, f_{n-1} \in \mathcal{F}$  their diagonal product  $(f_i)_{i < n} : X \rightarrow 2^n$  is surjective.

**Proposition 4.1.** *For any set  $X$  a function family  $\mathcal{F} \subset 2^X$  is independent if and only if it is  $\mathcal{B}^X$ -free in the clone  $2^X$ .*

*Proof.* To prove the “if” part, assume that  $\mathcal{F}$  is  $\mathcal{B}^X$ -free but not independent. Then there are pairwise distinct functions  $\xi_0, \dots, \xi_{n-1} \in \mathcal{F} \subset 2^X$  whose diagonal product  $\delta = \Delta_{i < n} \xi_i : X \rightarrow 2^n$ ,  $\delta : x \mapsto (\xi_i(x))_{i < n}$  is not surjective and hence the set  $F = 2^n \setminus \delta(X)$  is not empty. Let  $\alpha, \beta : 2^n \rightarrow 2$  be two algebraic operations on 2 defined by  $\alpha^{-1}(1) = F$  and  $\beta^{-1}(1) = \emptyset$ . They induce the algebraic operations  $\alpha^X, \beta^X : (2^X)^n \rightarrow 2^X$  on  $2^X$ . It follows that for the function  $\xi : n \rightarrow \mathcal{F}$ ,  $\xi : i \mapsto \xi_i$ , we get  $\alpha^X(\xi) = \beta^X(\xi)$ .

On the other hand, Since  $\alpha^X \neq \beta^X$ , there is a function  $\tilde{\xi} : n \rightarrow 2^X$  such that  $\alpha^X(\tilde{\xi}) \neq \beta^X(\tilde{\xi})$ . Now choose any function  $f : \mathcal{F} \rightarrow 2^X$  such that  $f|\xi(n) = \tilde{\xi} \circ \xi^{-1}$ . Since  $\mathcal{F}$  is  $\mathcal{B}^X$ -free, there is a function  $\bar{f} : \mathcal{B}^X(\mathcal{F}) \rightarrow 2^X$  such that  $\bar{f}(\gamma^X(\xi)) = \gamma^X(f \circ \xi)$  for any algebraic operation  $\gamma \in \mathcal{B}$  with  $S_\gamma = n$ . In particular,  $\bar{f}(\alpha^X(\xi)) = \alpha^X(f \circ \xi) = \alpha^X(\tilde{\xi}) \neq \beta^X(\tilde{\xi}) = \beta^X(f \circ \xi) = \bar{f}(\beta^X(\xi))$ , which contradicts  $\alpha^X(\xi) = \beta^X(\xi)$ . This contradiction completes the proof of the “if” part.

To prove the “only if” part, assume that  $\mathcal{F}$  is independent but not  $\mathcal{B}^X$ -free. By Proposition 3.4, there are two distinct operations  $\alpha, \beta \in \mathcal{B}$  such that  $S_\alpha = S = S_\beta$  for some finite set  $S \subset \omega$  and  $\alpha^X(\xi) = \beta^X(\xi)$  for some injective function  $\xi : S \rightarrow \mathcal{F}$ . The function  $\xi$  can be written in the form  $(\xi_i)_{i \in S}$  where  $\xi_i = \xi(i) \in \mathcal{F}$ . The independence of  $\mathcal{F}$  guarantees that the diagonal product  $(\xi_i)_{i \in S} : X \rightarrow 2^S$ ,  $\delta = (\xi_i)_{i \in S} : x \mapsto (\xi_i(x))_{i \in S}$  of these functions is surjective. Then  $\alpha \neq \beta$  implies  $\alpha \circ \delta \neq \beta \circ \delta$ , which yields a point  $x \in X$  such that  $\alpha \circ \delta(x) \neq \beta \circ \delta(x)$ . Now observe that

$$\alpha^X(\xi)(x) = \alpha((\xi_i(x))_{i \in S}) = \alpha \circ \delta(x) \neq \beta \circ \delta(x) = \beta((\xi_i(x))_{i \in S}) = \beta^X(\xi)(x)$$

which contradicts  $\alpha^X(\xi) = \beta^X(\xi)$ .  $\square$

By a classical Fichtenholtz-Katorovitch-Hausdorff Theorem 17.20 [6], the power-set  $\mathcal{P}(X)$  of each infinite set  $X$  contains an independent family  $\mathcal{F} \subset \mathcal{P}(X)$  of cardinality  $|\mathcal{F}| = |2^X| = |\mathcal{P}(X)|$ . Reformulating this result with help of Proposition 4.1, we get the following result:

**Corollary 4.2.** *For each infinite set  $X$  the power-clone  $2^X = (2^X, \mathcal{B}^X)$  contains a  $\mathcal{B}^X$ -free subset  $B \subset 2^X$  of cardinality  $|B| = |2^X|$ .*

In fact, Corollary 4.2 is a partial case of the following theorem, which is the main result of this paper.

**Theorem 4.3.** *Assume that a universal algebra  $\mathbb{A} = (A, \mathcal{A})$  has cardinality  $|A| \geq 2$ . For each infinite set  $X$  of cardinality  $|X| \geq |\mathcal{A}|$  the function algebra  $\mathbb{A}^X = (A^X, \mathcal{A}^X)$  contains an  $\mathcal{A}^X$ -free subset  $\mathcal{F} \subset A^X$  of cardinality  $|\mathcal{F}| = 2^{|X|}$ .*

*Proof.* Let  $\bar{\mathcal{A}}$  be the clone of the operation family  $\mathcal{A}$ . It has cardinality  $|\bar{\mathcal{A}}| \leq \max\{|\mathcal{A}|, \aleph_0\} \leq |X|$ . Since each  $\bar{\mathcal{A}}^X$ -free subset of  $A^X$  is  $\mathcal{A}^X$ -free, we lose no generality assuming that  $\mathcal{A}$  is a clone. In particular,  $\mathcal{A}$  is unital,  $\emptyset$ -regular and substitution-stable.

For each finite subset  $S \subset \omega$ , consider the family

$$\mathcal{T}_S = \{(\alpha, \beta, s) \in \mathcal{A} \times \mathcal{A} \times X^S : S_\alpha = S = S_\beta, \alpha \neq \beta\}$$

and observe that it has cardinality  $|\mathcal{T}_S| \leq |\mathcal{A} \times \mathcal{A} \times X^S| \leq |X|$ . Then the union  $\mathcal{T} = \bigcup_{S \in [\omega]^{<\omega}} \mathcal{T}_S$  where  $S$  runs over all finite subsets of  $\omega$  also has cardinality  $|\mathcal{T}| \leq |X|$  and hence admits an enumeration  $\mathcal{T} = \{(\alpha_x, \beta_x, s_x) : x \in X\}$  by points of the set  $X$ . By the definition of the family  $\mathcal{T}$ , for every  $x \in X$  the algebraic operations  $\alpha_x$  and  $\beta_x$  are distinct. Consequently, we can choose a function  $p_x : S_x \rightarrow A$  defined on the set  $S_x = S_{\alpha_x} = S_{\beta_x}$  such that  $\alpha_x(p_x) \neq \beta_x(p_x)$ .

Using Fichtenholtz-Katorovitch-Hausdorff Theorem 17.20 [6], fix an independent subfamily  $\mathcal{U} \subset \mathcal{P}(X)$  of cardinality  $|\mathcal{U}| = 2^{|X|}$ .

For each set  $U \in \mathcal{U}$  define a function  $f_U : X \rightarrow A$  assigning to a point  $x \in X$  the point  $p_x(i)$  where  $i$  is a unique point of the set  $s_x^{-1}(U) \subset U_x$  if this set is a singleton, and an arbitrary point of  $A$  otherwise.

We claim that the set  $\mathcal{F} = \{f_U\}_{U \in \mathcal{U}} \subset A^X$  has cardinality  $|\mathcal{F}| = 2^{|X|}$  and is  $\mathcal{A}^X$ -free.

**Claim 4.4.** *The function  $f : \mathcal{U} \rightarrow \mathcal{F}$ ,  $f : U \mapsto f_U$ , is bijective and hence  $|\mathcal{F}| = |\mathcal{U}| = 2^{|X|}$ .*

*Proof.* Given two distinct sets  $U, V \in \mathcal{U}$ , we should prove that  $f_U \neq f_V$ . By the unitality, the operation family  $\mathcal{A}$  contains the identity operation  $\text{id} : A^1 \rightarrow A$ ,  $\text{id} : (a) \mapsto a$ . Consider the embeddings

$$\sigma_0 : 1 \rightarrow 2 \quad \sigma_0 : 0 \mapsto 0 \quad \text{and} \quad \sigma_1 : 1 \rightarrow 2, \quad \sigma_1 : 0 \mapsto 1.$$

The substitution-stability of  $\mathcal{A}$  implies that the algebraic operations  $\alpha = \text{id} \circ \sigma_0^* : A^2 \rightarrow A$ ,  $\alpha : (a, b) \mapsto a$ , and  $\beta = \text{id} \circ \sigma_1^* : A^2 \rightarrow A$ ,  $\beta : (a, b) \mapsto b$ , belong to the operation family  $\mathcal{A}$ . It follows from  $|A| \geq 2$  that  $\alpha \neq \beta$ .

By the independence of  $\mathcal{U} \ni U, V$ , there is a function  $s : 2 \rightarrow X$  such that  $s(0) \in U \setminus V$  and  $s(1) \in V \setminus U$ . The triple  $(\alpha, \beta, s)$  belongs to the family  $\mathcal{T}_2 \subset \mathcal{T}$  and hence is equal to  $(\alpha_x, \beta_x, s_x)$  for some  $x \in X$ . Then  $p_x \in A^2$  is a function such that  $p_x(0) = \alpha_x(p_x) \neq \beta_x(p_x) = p_x(1)$ . The definition of the functions  $f_U$  and  $f_V$  guarantees that  $f_U(x) = p_x(0) \neq p_x(1) = f_V(x)$  and hence  $f_U \neq f_V$ .  $\square$

**Claim 4.5.** *The set  $\mathcal{F}$  is  $\mathcal{A}$ -free.*

*Proof.* Assuming that  $\mathcal{F}$  is not  $\mathcal{A}$ -free and applying Proposition 3.4, find a finite subset  $S \subset \omega$ , an injective function  $\xi : S \rightarrow \mathcal{F}$  and two distinct algebraic operations  $\alpha, \beta \in \mathcal{A}$  such that  $S_\alpha = S = S_\beta$  and  $\alpha^X(\xi) = \beta^X(\xi)$ .

For every  $i \in S$  find a set  $U_i \in \mathcal{U}$  such that  $\xi(i) = f_{U_i}$ . The independence of the family  $\mathcal{U}$  guarantees the existence of a function  $s : S \rightarrow X$  such that  $s(i) \in U_i \cap \bigcap_{j \in S \setminus \{i\}} (X \setminus U_j)$ . It follows that  $s^{-1}(U_i) = \{i\}$  for each  $i \in S$ .

The triple  $(\alpha, \beta, s)$  belongs to the family  $\mathcal{T}_S \subset \mathcal{T}$  and hence is equal to  $(\alpha_x, \beta_x, s_x)$  for some point  $x \in X$ . For every  $i \in S$  the definition of the function  $f_{U_i}$  guarantees that  $f_{U_i}(x) = p_x(i)$ . Let  $\delta_x : A^X \rightarrow A$ ,  $\delta_x : g \mapsto g(x)$ , denote the  $x$ -th coordinate projection. We claim that  $\delta_x \circ \xi = p_x$ . Indeed, for each  $i \in S$  we get  $\delta_x \circ \xi(i) = \delta_x(f_{U_i}) = f_{U_i}(x) = p_x(i)$ .

Applying the function  $\delta_x$  to the equality  $\alpha^X(\xi) = \beta^X(\xi)$ , we get  $\delta_x \circ \alpha^X(\xi) = \delta_x \circ \beta^X(\xi)$ . On the other hand,  $\delta_x \circ \alpha^X(\xi) = \alpha(\delta_x \circ \xi) = \alpha(p_x) \neq \beta(p_x) = \beta(\delta_x \circ \xi) = \delta_x \circ \beta^X(\xi)$ , which is a desired contradiction.  $\square$

$\square$

**Problem 4.6.** *Let  $\mathbb{A} = (A, \mathcal{A})$  be a universal algebra with  $|A| \geq 2$ ,  $X$  be an infinite set of cardinality  $|X| \geq |\mathcal{A}|$ , and  $\mathcal{F} \subset A^X$  be a maximal  $\mathcal{A}^X$ -free subset. Is  $|\mathcal{F}| \geq |2^X|$ ?*

**Problem 4.7.** *Let  $\mathbb{A} = (A, \mathcal{A})$  be a universal algebra such that the operator of  $\mathcal{A}$ -hull has the MacLane-Steinitz exchange property. Is each  $\mathcal{A}$ -independent subset of  $A$   $\mathcal{A}$ -free?*

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